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Mathematical Modelling of Human Population Growth

Dr. Neeta Mazumdar

Associate Professor, Department of Mathematics, Government College of Arts,Commerce & Science, Quepem, Goa,India

Abstract: **This work presents models and methods that have been used in producing forecasts of population growth. The work is intended to emphasize the reliability bounds of the model forecasts. Leslie model is analysed. The Leslie modelling approach involves the use of current trends in mortality, fertility, migration and emigration. The model treats population divided in age groups and the model is given as a recursive system. Other group of models are based on straightforward extrapolation of census data.**

The work presents the basics of Leslie type modelling ,the latter model is also analysed from model reliability point of view.

Keywords: **Exponential model, Euler-Lotka equation, Logistic model, Leslie Model, Demography, Population census data.**

1. INTRODUCTION

Population has been a controversial subject for ages. Charles Darwin ones said, ‘ In the struggle for life number gives the best insurance to win’. Every government and collective sector always require accurate idea about the future size of various entities like population, resources, demands, consumptions and so on, for their planning activities. To obtain this information, the behaviour of the connected variables is analysed based on the previous data. Using the conclusions drawn from the analysis future projections of the variables is made.

At present, there exists two major examples in statistics namely Conventional and Bayesian methodology in the interest of data analysis. The use of Bayesian methodology in the field of data analysis is relatively new and has found major support in last two decades from the people belonging to various disciplines. Apparently the main reason behind the increasing support is its flexibility and generality that allows it to deal with the complicate situations.

There is enormous concern about the consequences of human population growth for social, environment and economic development. Intensifying all these problems is population growth. Globally, the growth rate of the human population has been declining since peaking in 1962 and 1963 at 2.20% per annum. In 2009, the estimated annual growth rate was 1.1%. The last 100 years have seen a rapid increase in population due to medical advances and massive increase in agricultural productivity, made possible by the Green Revolution. The actual annual growth in the number of humans fell from its peak of 88.0 million in 1989, to a low of 73.9 million in 2003, after which it rose again to 75.2 million in 2006. Since then, annual growth has declined. In 2009, the human population increased by 74.6 million, which is projected to fall steadily to about 41 million per annum in 2050, at which time the population will be increased to about 9.2 billion. Each region of the globe has seen great reductions in growth rate in recent decades, though growth rates remain above 2% in some countries of the Middle East and Sub-Saharan Africa, and also in South Asia, Southeast Asia, and Latin America. Some countries experience negative population growth, especially in Eastern Europe mainly due to low fertility rates, high death rates and emigration. In Southern Africa, growth is slowing due to the high number of HIV-related deaths. Some European countries might also encounter negative population growth. Japan's population began decreasing in 2005. The United Nations Population Division expects world population to peak at over 10 billion at the end of the 21st century. In recent times there have been big developments in analysis of population.

1.1 Definitions and estimation of population growth rate

The summary of parameter of any trends in population density or in abundance is known as population growth rate. In case of whether density and abundance are increasing or de-creasing which inform as how fast they changes in terms of population growth rate. Population growth rate normally describes the per capita growth of population $\frac{1}{N}\frac{dN}{dt}$ (In the absence of limitations to growth, food and territorial).

If $N = N(t)$ is the population at time $t \geq 0$, and initial population size is $N(0) = N_0$ at t=0, then $\lambda = \frac{N_{t+1}}{N_t}$ $or\ r = \log_e \lambda$ where r is growth rate and λ is finite growth rate respectively. In order to estimate population growth rate we normally use either population census data from period of time or from demographical data.

1.2 Importance in projection of future population sizes

To predict any future projection of population sizes of a given place it is important to know the population growth rate. Without density dependence, the population growth then becomes exponential. By using Euler-Lotka equation, the growth rate can be calculated from demographic data for existing population. The projections of future population are normally based on the present population. As we know population must to be articulated in order to make useful projections possible.

1.3 Historical background of population study

The pivotal study of population growth rate has been recognised for long time.

In (1798), Thomas Malthus wrote a paper on ‘ An Essay on the Principle of Population’. Earlier, in the late seventeenth century the mortality table with mathematical analyses were analysed by Huygens and later Buffon among others. Cole suggested that Newton outstandingly failed to comprehend that the basic concept of expectancy was a function of age, the mathematical dependence of population growth rate was basically on age-specific birth rates and death rates, which were based on these two principles, that of mortality and the fertility, which once they have been established for a certain place, make it easy to resolve all the questions which one could propose. Verhulst proposed the logistic equation, for population growth. In this paper the relationship between population growth rate and population model is examined. The above ideas will help us to study the identification of good model for population growth.

2. POPULATION STUDY

Human population have become the subject of changes in the number and age-structure.

These changes normally take place through the processes of birth, deaths and the counter balance between immigration and emigration. The development of any country is based on a collective statistical information data. These has assisted many countries to collect data available about its current populations, some of these are enormous in less developed countries. During the seventeenth century, however, men became interested in the study of human population purely from scientific point of view. The first person was an Englishman, John Graunt (1620-1674). He introduced the first life table and studies of population of London in some detail. Many then followed his footsteps about the study of human population, Later Thomas Malthus in 1798, in his work ‘An Essay on the Principle of Population’, painted a pessimistic picture of the future. He argued that the geometrical growth of the human population would soon outgrow the arithmetic progression of the world’s population, leaving the world’s population in dire state.

Let $N(t)$ be the population size of number people at time t and $n(t)$ be the concentration of the rate limiting substrate, then we have the simple hypothesis,

$$\frac{dN}{dt} = K(n)N(t) \qquad (1)$$

where $k(n)$ is the specific growth rate of the population.

Let us consider the following scenario: $N(t)$ is the population size, $n(t)$ be the concentration within an ecosystem.

When population size $N(t)$ consumes more of $n(t)$, the rate of change of concentration would be less. The substrate is consumed by the population size and this was proposed by Monod's model, which defines the relation between the growth rate and the concentration of the rate-limiting substrate.

$$\frac{dn}{dt} = -\frac{1}{Y} K(n) N \qquad (2)$$

where Y is called the yield coefficient. From (1) and (2)

$$N(t) + Yn(t) = N(0) + Yn(0) = \bar{n} \;\; (say) \qquad (3)$$

$so \quad (1)\, gives$

$$\frac{dN}{dt} = N(t) K \left[\frac{\bar{n} - N(t)}{Y} \right] \qquad (4)$$

Integrating both the sides we get,

$$\log[N]_{N_0}^{N} = Kt \qquad (5)$$

$which \;\; gives,$

$$N = N_0 e^{Kt} \qquad (6)$$

Where, K is the productivity rate, the (constant) ratio of growth rate to population, N_0 is the initial population when time t=0.

Basically all these cannot be done without demography method or process.

2.1 Scenarios of Future Population

What can we say about the future of world population? The simplest projection is to assume that current fertility rates continue to exist indefinitely into the future. Since the fertility rate is greater than one, population increases exponentially according to the formula $N(1+n)$, where N is the initial size of the population and n is the rate of increase; if N is 1 million and n is 2, the population will be 2 million in 34.4 years, 4 million in 69 years and so forth. This is the Malthusian method of projecting population growth. This method shows that within a readily predictable amount of time there will be more human beings than there are atoms in the universe!

2.2 Conservative nature of assumptions

In addition, even these assumptions probably understate the possible variance contained in the current, really rather crude models. The Low assumption is based on a fertility rate of 1.7 and the High on a fertility rate of 2.5. But there's no reason at all to believe that these are the real low and high limits, since the current fertility rate in India is 2.51 and in East Africa is 6.1. Applying these numbers to (6) gives a truly titanic swing, everywhere from 1 to 200 billion.

2.2.1 Replacement rate immediately

The youthful nature of the earth's population means that the population will continue to increase for a while even if the fertility rate falls to replacement rate immediately, Replacement rate being just over two children per woman, on the average. This is because the number of women of childbearing age will continue to increase for several decades into the future. According to one illustrative UN projection, an immediate fall of fertility to the replacement rate would mean that the population would continue to increase until about 2100 and then stabilize at 8.4 billion.

2.2.2 Replacement rate and spread of estimates

This is an illustration of the importance of sensitive dependence on initial conditions. If instead of the replacement rate of 2.06, we substitute a rate of 1.96, one-tenth child per woman fewer, or five percent less than 2.06 the population would be 5.4 billion in 2150 and drop thereafter; if we assume a rate of 2.16-one-17 tenth child per woman more, or five percent more than 2.06 the population in 2150 is over 20 billion.

2.3 Implicit negative assumptions

These projections are basically linear projections and they depend on implicit negative assumptions, by which it means, there are assumptions that nothing will change that will affect fertility. Let's name a few of these implicit assumptions: there will be linear economic development, the absence of epidemic disease, the absence of large-scale war, no basic changes in agricultural productivity, no breakthroughs in energy technology, and so on and so forth. This very simple analysis throws in high relief the things that really are relevant to the population/resource relationship.

2.3.1 The demographic trap

One fear is that the fall in fertility depends on economic development, on the movement away from the high-family, largely rural society to the urbanized, small-family model of modern society. But that movement depends on the accumulation of wealth, which inturns depends on the possibility of savings, $Y = C + S$, where C represents the account and S represents the accumulation of the saving. But if population is so high that just keeping level with current consumption takes all the economic activity the country produces, no savings are possible and the movement to industrialization is checked. Or, to put it in relative rather than absolute terms, the amount of C will vary with the youthfulness of the population, and will make the accumulation of S that much more difficult; thus prolonging the time it takes to accomplish the demographic transition and leading to larger populations. This shows that zero population growth rates will eventually be achieved, because any long-term growth rate greater than zero (that is, any 'n' greater than zero in the formula $N(1+n)$) implies an exponential growth rate and a population that will grow until all the matter in the universe has been converted into human tissue, the only question is how long this process will take. But this limiting case is, again, not really useful for policy analysis. The number of variables and the complexities of their interactions increase rather than decreases in time, so that, at least in terms of the present level of science, the big questions of sustainability are unanswerable, and the attempts to answer it so far have not usually worked out. In these circumstances, the best we can do is to advance science; to find out what the relevant factors are and to begin to see how they fit together.

The only way that the future population of the earth can be predicted is if it is made to come out a certain way. But even this is not certain, because it depends not only on making a certain policy on a world scale but on being sure we stick to it. Chinese fertility rates have been undergoing wild swings precisely because the government has been trying to exercise conscious control over fertility, but the policy keeps changing; so the result of human intervention in fertility is to make the swings in fertility rates more dramatic, and probably more unpredictable, than they were before massive intervention began.

Nonetheless, the entire trend of human history is in favour of more control rather than less.

Malthus believed that population would go on increasing until it was checked by famine, disease or war; because he believed that the human instinct for reproduction-for sex-could not be subjected to conscious control. This was fundamentally wrong, as the change in fertility rates over the past few decades' shows. Agriculture is the exertion of conscious control over the food supply, and in the past two centuries, pace Malthus, we have exerted conscious control over the size of our own families.

Leslie Matrix Population Model has contributed to population demographic model and it has been that found to be most useful in determining population growth. Matrix population models have been transformed into useful analysis of predicting population growth. In 1959, Leslie came out with modified form of projection of matrix that was allowed for the effect of the existence of other population members on the population growth.

2.4 Projections using the Leslie model

In order to understand the dynamics principle of population growth, we need to project a matrix model (Leslie). Appraisal will be made in the light of vital rates, which will depend on continuous survivorship and fertility functions.

Definition 2.4.1: The survivorship function is the chance of an individual surviving from birth to age x, and it can be rescaled to given a number of survivors from the initial cohort.

It is mathematically denoted by $l(x)$,where $l(x)$ is probability of the survivors.

Definition 2.4.2: The fertility function is the expected number of offspring (female off-spring) per individuals of age x at a unit time, and it is denoted by $m(x)$.

3. HUMAN POPULATION MODELS

In population growth many process in biology and related to other fields illustrate S shaped growth. These curves were well modelled with the logistic growth function which was first introduced by Verhulst in 1845. This logistic curve has been placed under severe criticism where the system is not remarkable, on other hand this has been proved useful in a wide range of phenomena. In recent times, Young assessed and distinguished growth curves used for concerning detail forecasting, containing the logistic function. All this was applied in case of single growth process managing in seclusion. In case of human system the carrying capacity is always restricted by the contemporary level of technology, which is uncertain. Several models have been used to determined the number of population growth of estimated country.

3.1Leslie model

The Leslie matrix population model is a discrete (i.e., time goes in steps as opposed to continously) and age dependent model (construction of the model consider only age). The Leslie matrix population model is widely used in population ecology and demography in order to determine the growth of the population, as well as the age distribution within the population over time. The aftermath of population inconstancy is mostly afflicted by the density dependency and stochasticity. P. H. Leslie put in place a suggestion of projection matrix model on the effect of population growth onto the other population members. In 1966, Pollard studied into the stochastic action towards his model. The diagram below shown the discretization of the age classes and time, where class i corresponds to ages $i-1 \le x \le i$.

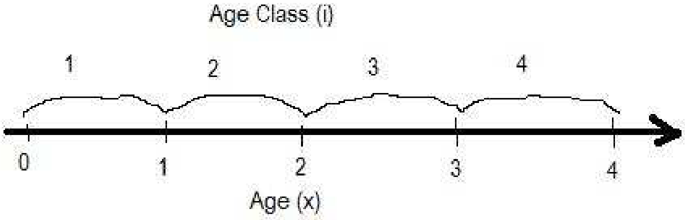

Figure 1: Discrete age class i and continuous age x.

3.1.1 Density Dependence in the Leslie Model

Leslie wrote a lot of papers in (1945, 1948, 1959) deal with the density as tally of all individuals in the population, no matter what the age. The population size is given by

$$N(t) = \sum n_i(t) \qquad (7)$$

Where, $n(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ . \\ . \\ . \\ n_w(t) \end{bmatrix}$ and $n_i(t)$ =number of females of age i

He defined the quantity of his postulating of the population density with each time interval of the different age group as

$$q(t) = 1 + aN(t) \qquad (8)$$

where a is the density parameter is given by

$$a = \frac{\lambda - 1}{K}$$

When a = 0 and a < 0, there will be no population density and with negative entries in the model respectively. In case of this, a must always be greater than 0 with the condition that all age class in consideration of $q(t)$) must also be greater than 1. If the population is less than the carrying capacity K then the per capita growth rate is positive and the population increases, and after the population become stable and the total size of the population always remain constant. The $q(t)$ values are the diagonal elements of thematrix Q

$$Q(t) = \begin{bmatrix} q_1(t) & 0 & 0 & 0 \\ 0 & q_2(t) & 0 & 0 \\ 0 & 0 & q_3(t) & 0 \\ & & & \\ 0 & 0 & 0 & q_w(t) \end{bmatrix} \qquad (9)$$

We must note that the number of individuals in the age groups at time t can now be mapped to time $t+1$ as

$$n(t+1) = AQ^{-1}n(t) \qquad (10)$$

He introduced time-lag in 1959 by basic model. When these lags are then taken into consideration, the age groups in the population for each class i one has

$$q_i(t) = 1 + aN(t-i-1) + bN(t) \qquad (11)$$

where b is the effect of density at birth on the probability survival at the later stage.

Both a and b are > 0, and their magnitude is $\frac{b}{b+a}$

Always the elements in the projection matrix are then divided into two depending on

• the size of the current population at time t , and

• the size of the population at time $t-i-1$, which is the commencement of theinterval where individuals were currently born of age i .

Subsequently, sure number of projection will arrive at stability at time $\tau, q(t) = \lambda \ \forall i.$

Thus

$$A(\tau) = AQ^{-1}(\tau) = \lambda^{-1}A \qquad (12)$$

accordingly the matrix becomes

$$A(T)=\begin{bmatrix} \frac{F_1}{\lambda} & \frac{F_2}{\lambda} & \frac{F_3}{\lambda} \cdots\cdots\cdots & \frac{F_{w-1}}{\lambda} & \frac{F_w}{\lambda} \\ \frac{P_1}{\lambda} & 0 & 0 \cdots\cdots\cdots & 0 & 0 \\ 0 & \frac{P_2}{\lambda} & 0 \cdots\cdots\cdots & 0 & 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & 0 & \frac{P_{w-1}}{\lambda} & 0 \end{bmatrix} \qquad (13)$$

In population dynamics there is brainwork to decline the vital rate due to competition, and other factors that affect the population growth. The entries of any elements of the density-dependence matrix are either compensatory, over compensatory or dispensatory.

This entry then becomes from the results of structured populations:

$$N(t+1)=f(N)=g(N)N \qquad (14)$$

where the function $g(N)$ is the rate per-capita, and $f(N)$ is the recruitment function.

The total population is given by

$$N(t)=\sum n_i(t) \qquad (15)$$

When $N>0, \frac{dg(N)}{dN}>0$,

Furthermore $g(N)$ is said to make dispensation, and if $\frac{dg(N)}{dN}\le 0, \ \frac{df(N)}{dN}\ge 0 \ and \ \lim_{n\to\alpha} f(N)=C>0.$

on other hand $g(N)$ becomes compensatory. When

$\lim_{n\to\alpha} f(N)=0$ in all these $g(N)$ displays overcompensation.

3.1.2 Stochastic in the model

In stochastic model, F_i is defined to be the probability that a female in age group i at the time t will be give birth to a single daughter during the time interval (t, t + 1) and the this daughter will be alive at time t+1 to be enumerated in age group 0−t. The stochastic process is consistently estimated on the condition that if there is random variation over time, and this is in line with the Leslie model, force A to be A_t (A is now a function of time). In case of the variation, there are physical or biological factors in the ecosystem.

As a results of this, we can group these into two of stochasticity:

• Environment, • Demographic stochastic

In developing stochastic projection model there is one basic step that we need to consider.

Absorption of variance to change from deterministic to stochastic and the matrix model will be given by

$$n(t+1)=A_t n(t)$$

where A_t is the column stochastic transition matrix. A_t is homogenous when the environment is constant otherwise inhomogeneous.

Leslie model can be constructed by the expectation of the variable $n_t(t)$, for instance when we consider the fact that the number of females of the age i at time t at fixed Pi and Fi , and mutually independent, thus $n_t(t+1)$ of it binomial variable B($n_t(t), F_t)$) is given

By

$$e_{t+1} = A_t e^t \qquad (16)$$

Thus,

$$\begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ . \\ e_n \end{bmatrix}(t+1) = \begin{bmatrix} F_1 & F_2 & . & . & . & F_w \\ P_1 & & & & & \\ 0 & P_2 & & & & \\ . & & P_3 & & & \\ . & & & & & \\ 0 & . & . & . & P_{w-1} . . & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ . \\ e_w \end{bmatrix}(t) \qquad (17)$$

where,

$$e(t) = \begin{bmatrix} e_1 \\ e_2 \\ . \\ . \\ . \\ e_w \end{bmatrix} \quad and \quad e_i(t) = \text{expected number of females of age } i.$$

3.1.3 Example

Illustration: The following example illustrates production of salmon population by using

Lesslie's Model. There are specification that one to needs for the stochastic process.

We let the stochastic process to be $y(t)$ and also a good year to be $y(t)$ equals to $1.5(say)$ and in a bad year $y(t)$ equals to $0.43(say)$. Then allow the good and bad year to occur randomly by flipping a coin, and independently with probability 0.6. Therefore, **A** can be written as

$$A_t = \begin{bmatrix} 0 & 4y(t) & 5y(t) \\ 0.53 & 0 & 0 \\ 0 & 0.22 & 0 \end{bmatrix}$$

The above example illustrates how the production of salmon population takes place. From

(16) where A_t is randomly chosen with $y(t) = \{1.5, 0.43\}$ with equal probability. Let suppose the initial population vector is $e_0 = [10 \ 10 \ 10]$. That is the population age distribution vectors for first ten years. We generate a sequence of matrix equations to find the production of salmon population as follows

$$e_1 = A_t e_0$$
$$e_2 = A_t e_1$$
$$e_3 = A_t e_2$$

Graphs can be drawn to show production of salmon population at different levels.

3.2 Conclusion

The paper demonstrates how the Leslie matrix provides understanding of the mathematics behind the parameters in the matrix. The major outcome was that the matrix depends only on the fertility and survival rate. In future when we get statistical data trends on mortality, fertility and immigration in population growth, it is appropriate to apply these factors to the population of the country in consecutive years in the future, starting with the population size and structure being put in place. Projection and its reliability bounds provides forecast for the future, which help with analysis.

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